Dynamic scaling of growing interfaces: The Kardar–Parisi–Zhang equation

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A model for the dynamics of fluctuating, far-from-equilibrium interfaces was proposed by M. Kardar, G. Parisi, and Y.-C. Zhang [Phys. Rev. Lett. **56** (1986)], and is briefly discussed here. Novel aspects of the development are highlighted, and their impact on statistical physics is described. Connections to other physical systems and recent advances are commented on.

Dynamic, fluctuating interfaces are ubiquitous in far-from-equilibrium systems. Examples include the edge of an outwardly expanding bacterial colony in a petri dish [1], the flame front of an ignited sheet of paper [2], and the interfaces between defect-rich and defect-poor phases in turbulent liquid crystals [3]—the latter of which is presented in Fig. 1. If the interface is described by a height field $h(x,t)$ above the x-axis, where $x \in [0, L]$, then we quantify the average height $\bar{h}(t)$ and interface width $w(t, L)$, respectively, as

$$\bar{h}(t) := \frac{1}{L}\int_0^L h(x,t)\ \mathrm{d}x \tag{1}$$

and

$$\big(w(t,L)\big)^2 := \frac{1}{L}\int_0^L \Big(h(x,t) - \bar{h}(t)\Big)^2\ \mathrm{d}x\ . \tag{2}$$

Following the well-known scaling hypothesis in statistical physics, we expect the interface width to scale as

$$w(t,L) \sim L^{\chi}\, F(t/L^{z})\ , \tag{3}$$

where F is some universal function, χ is the roughening exponent, and z is the dynamic exponent. Remarkably, the scaling exponents are identical in seemingly unrelated physical systems—including the bacterial colonies, ignited paper, and liquid crystals mentioned earlier—and are experimentally measured to be

$$\chi^{\mathrm{exp}} = \frac{1}{2} \qquad \text{and} \qquad z^{\mathrm{exp}} = \frac{3}{2}\ . \tag{4}$$

Such results are a hallmark of *universality* in the dynamics of physical systems.

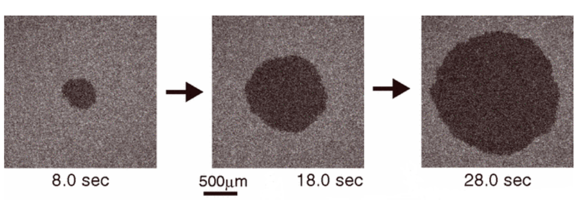

FIG. 1. Experimental images of turbulent liquid crystals, highlighting the growth of a defect-rich phase (dark grey) in a defect-poor phase (light grey). The dynamic, fluctuating interface between the two phases exhibits the scaling behavior given in Eqs. (3) and (4). Image reproduced from Ref. [3].

A major objective of modern statistical physics is to describe universal behavior in seemingly unrelated physical systems, even those far from equilibrium. To this end, we recognize the height $h(\boldsymbol{x}, t)$ as an order parameter field. Here and henceforth, $\boldsymbol{x} \in \mathbb{R}^d$ describes a d-dimensional surface, about which an interface fluctuates in $d+1$ spatial dimensions ($d = 1$ in the experimental examples discussed previously). Moreover, we recognize interfacial dynamics are invariant to the choice of the origin of the height field, such that only derivatives of h enter our description of the interface. The lowest-order effective Hamiltonian of the interface is then given by

$$\mathcal{H}_{\mathrm{eff}} = \int \mathrm{d}^d\boldsymbol{x} \left(\frac{\nu}{2}\left(\boldsymbol{\nabla} h\right)^2\right)\ . \tag{5}$$

We note that while higher-order terms could be included in Eq. (5), they contain more spatial derivatives and are expected to be less relevant in describing universal behavior. By drawing analogy between the dynamics of an order parameter field and those of an overdamped point particle in a surrounding medium [4], we expect a height field whose energetics are governed by Eq. (5) to evolve in time according to

$$\frac{\partial h(\boldsymbol{x},t)}{\partial t} = \nu \nabla^2 h + \eta(\boldsymbol{x},t)\ , \tag{6}$$

known as the Edwards–Wilkinson (EW) equation [5]. In Eq. (6), $\eta(\boldsymbol{x},t)$ is the thermal noise, assumed to be Gaussian with moments $\langle \eta(\boldsymbol{x},t)\rangle = 0$ and

$$\langle \eta(\boldsymbol{x},t)\,\eta(\boldsymbol{x}',t')\rangle = 2D\delta^d(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t')\ . \tag{7}$$

As the EW equation (6) is linear, it can be solved exactly, for which the dynamical exponents are given by

$$\chi^{\mathrm{EW}} = \frac{2-d}{2} \qquad \text{and} \qquad z^{\mathrm{EW}} = 2\ . \tag{8}$$

Importantly, for a $d = 1$ dimensional interface, we find $\chi^{\mathrm{EW}}_{(d=1)} = 1/2$ and $z^{\mathrm{EW}}_{(d=1)} = 2$; the latter disagrees with experimental measurements [cf. Eq. (4)]. Moreover, we argued previously that introducing additional terms in the effective Hamiltonian (5) is not expected to alter the predicted scaling behavior (8). Thus, a fundamental question arises: How does one explain experimental observations of universal interfacial dynamics?

The discrepancy between theory and experiments in the scaling of interfaces was resolved in the seminal work by M. Kardar, G. Parisi, and Y.-C. Zhang in 1986 [6]. Crucially, the authors recognized that when (i) interface growth is everywhere locally orthogonal to the interface, and (ii) the interface is described by a vertical displacement above some surface, then geometric effects need to be taken into account. For example, consider a $d = 1$ dimensional interface with constant slope, as shown in Fig. 2. If there is a flux of material $v\delta t$ normal to the surface, then simple geometry reveals the change in vertical height δh is given by $\delta h = v\delta t\sqrt{1 + (\partial_x h)^2}$. When the slope is small ($\partial_x h \ll 1$), we find $\delta h/\delta t \approx v[1+\frac{1}{2}(\partial_x h)^2]$. Generalizing this result to a d-dimensional interface, we find

$$\frac{\delta h}{\delta t} = v\left(1 + \frac{1}{2}(\boldsymbol{\nabla} h)^2\right) . \tag{9}$$

Now recognizing we can describe the interface in a moving frame that absorbs the constant v term in Eq. (9), we include the gradient squared term in the EW equation (6) to arrive at the celebrated Kardar–Parisi–Zhang (KPZ) equation—given by

$$\frac{\partial h(\boldsymbol{x}, t)}{\partial t} = \nu\nabla^2 h + \frac{\lambda}{2}(\boldsymbol{\nabla} h)^2 + \eta(\boldsymbol{x}, t) . \tag{10}$$

In Eq. (10), ν is a surface tension that smooths the interface, while λ captures the nonlinear growth normal to the interface; the noise $\eta(\boldsymbol{x}, t)$ is identical to that in Eq. (6). Importantly, the nonlinear term in the KPZ equation (10) cannot be obtained from the variation of an effective Hamiltonian—and is thus a nontrivial step forward in the description of out-of-equilibrium fluctuating interfaces.

To determine the scaling behavior of the KPZ equation, we begin by prescribing the position $\boldsymbol{x}$, time t, and height h to scale as

$$\boldsymbol{x}' = b^{-1}\boldsymbol{x} , \qquad t' = b^{-z}t , \qquad \text{and} \qquad h' = b^{-\chi}h . \tag{11}$$

By substituting Eq. (11) into Eq. (10) and seeking an equation invariant to the rescaling, we find the tension

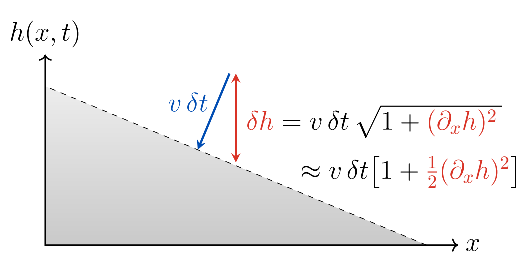

FIG. 2. Schematic showing the origin of the nonlinear term in the KPZ equation (10). For a flux of material $v\delta t$ normal to the interface (blue), the height field (red) changes by $\delta h = v\,\delta t\sqrt{1 + (\partial_x h)^2}$, which—when the slope is small—can be approximated as $\delta h \approx v\,\delta t\,[1 + \frac{1}{2}(\partial_x h)^2]$.

ν, diffusion constant D, and nonlinear parameter λ scale as

$$\nu' = b^{z-2}\nu , \qquad D' = b^{z-d-2\chi}D , \qquad \lambda' = b^{\chi+z-2}\lambda . \tag{12}$$

In the vicinity of the EW fixed point, $\nu' = \nu$ and $D' = D$, for which the scaling exponents are given by Eq. (8). At this fixed point, however, the nonlinear parameter scales as $\lambda' = b^{(2-d)/2}\lambda$—relevant under rescaling when $d = 1$, and marginal when $d = 2$. We thus expect to find qualitatively different physics between the EW and KPZ equations when $d = 1$, and possibly also when $d = 2$.

To systematically calculate how the paramters ν, D, and λ rescale, we employ the dynamic renormalization group (RG). The flow equations can be obtained via two different procedures. With the first method, employed in Refs. [7] and [8] (see also Ref. [9]), one takes the Fourier transform of Eq. (10) and uses an iterative procedure to find the flow equations to the appropriate order. In the second method, carried out in Ref. [10], one elevates the KPZ equation to an exponent via an auxiliary field, obtains the action, and then uses the Martin–Siggia–Rose procedure to calculate the flow equations. Both methods yield identical results; the flow equations at the one-loop level are given by

$$\frac{\mathrm{d}\nu}{\mathrm{d}\ell} = \nu\left[z - 2 + K_d\,\frac{\lambda^2 D}{\nu^3}\,\frac{2-d}{4d}\right] , \tag{13}$$

$$\frac{\mathrm{d}D}{\mathrm{d}\ell} = D\left[z - d - 2\chi + K_d\,\frac{\lambda^2 D}{4\nu^3}\right] , \tag{14}$$

and

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\ell} = \lambda\left[\chi + z - 2\right] , \tag{15}$$

where K_d is the surface area of the d-dimensinal unit sphere divided by $(2\pi)^d$. Equations (13) and (14) reveal the one-loop correction couples all three parameters through the quantity $\lambda^2 D/\nu^3$, while Eq. (15) shows the flow equation for λ is unaltered [cf. Eq. (12)]. Once again assuming we are at the EW fixed point (ν_*, D_*), where $(\mathrm{d}\nu/\mathrm{d}\ell)|_{(\nu_*,D_*)} = 0$ and $(\mathrm{d}D/\mathrm{d}\ell)|_{(\nu_*,D_*)} = 0$, we find all three flow equations can be combined into a single flow equation for the nonlinear parameter λ—given by

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\ell} = \frac{2-d}{2}\lambda + K_d\,\frac{2d-3}{4d}\,\frac{D_*}{\nu_*^3}\,\lambda^3 . \tag{16}$$

As we will now discuss in detail, Eq. (16) indicates λ can be relevant in any dimension, in which case the nonlinear term in the KPZ equation leads to new scaling behaviors. Moreover, as higher-order nonlinear terms in a Langevin description of a fluctuating interface will contain more spatial derivatives, they are expected to be less relevant under the RG flow and thus will not affect the long-time, long-wavelength dynamics. The KPZ equation thus brings about a new universality class.

In $d = 1$ dimensions, the flow of λ is given by $\mathrm{d}\lambda/\mathrm{d}\ell = \lambda/2 - \lambda^3 K_1 D_*/(4\nu_*^3)$, and there are two solutions λ_* where $\mathrm{d}\lambda/\mathrm{d}\ell = 0$. The solution $\lambda_* = 0$ is unstable, indicating an initially small value of λ will increase under the RG (see Fig. 3). On the other hand, the solution $\lambda_* = \sqrt{2\nu_*^3/(K_1 D_*)}$ is stable. The KPZ scaling exponents at the stable fixed point are found to be

$$\chi^{\mathrm{KPZ}}_{(d=1)} = \frac{1}{2} \qquad \text{and} \qquad z^{\mathrm{KPZ}}_{(d=1)} = \frac{3}{2}\,, \tag{17}$$

which are significant for two reasons. First, the RG KPZ exponents (17) agree with the experimentally measured exponents (4)—implying the nonlinear term in the KPZ equation leads to a new universality class in one dimension. Second, in $d = 1$ dimensions only, the KPZ equation can be solved exactly, with the stationary probability distribution of an interface height h given by [4, 9]

$$\mathcal{P}[h(x,t)] = \frac{1}{Z} \exp\left\{ -\frac{\nu}{2D} \int \mathrm{d}x \left[\left(\partial_x h\right)^2 \right] \right\}. \tag{18}$$

Remarkably, the exact scaling exponents of the one-dimensional KPZ solution (18) are identical to those from the one-loop RG analysis (17).

In Eq. (16), $d = 2$ is the critical dimension, for which there is no linear term. The nonlinear parameter λ flows according to $\mathrm{d}\lambda/\mathrm{d}\ell = \lambda^3 K_2 D_*/(8\nu_*^3)$. Here, $\lambda_* = 0$ is the only fixed point, and is unstable. An initially nonzero value of λ will increase under the RG and flow to some strong-coupling limit, and the nonlinearity in the KPZ equation is again relevant in two dimensions. Unfortunately, however, an RG analysis does not reveal what this limit is, and there is no exact solution for the KPZ equation in $d = 2$ dimensions. Accordingly, the KPZ scaling exponents can only be calculated numerically.

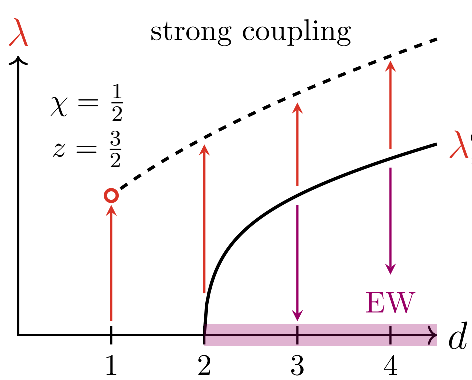

FIG. 3. Flow diagram of the parameter λ in different dimensions d, adapted from Refs. [9, 10]. As discussed in the text, λ is relevant in $d = 1$ and $d = 2$ dimensions. For $d \geq 3$, there is a critical value of λ above which it is relevant, and below which it is irrelevant. In the former case, the system flows to a strong-coupling limit where the interface is rough, while in the latter case the system flows to the EW equation, where $\lambda = 0$ and the interface is smooth.

In $d \geq 3$ dimensions, we again find two solutions λ_* where $\mathrm{d}\lambda/\mathrm{d}\ell = 0$. Here, the $\lambda_* = 0$ solution is stable, such that a system with an initially small value of λ will flow to the EW fixed point where $\lambda = 0$. An unstable fixed point exists at $\lambda_*^{\mathrm{c}} = \sqrt{2(d-2)\nu_*^3/[K_d D_*(2d-3)]}$, indicating that λ flows away from λ_*^{c} under the RG. Accordingly, the curve $\lambda = \lambda_*^{\mathrm{c}}$ serves as a separatrix for $d \geq 2$: if $\lambda < \lambda_*^{\mathrm{c}}$, the system flows to the EW fixed point and the nonlinearity is irrelevant, while if $\lambda > \lambda_*^{\mathrm{c}}$ then the system flows to a strong-coupling limit where the nonlinearity is relevant (see Fig. 3). Though the strong-coupling limit cannot be accessed via the RG, and no exact solution is known for $d \geq 3$, we recognize that $\chi > 0$ in the strong-coupling limit and $\chi < 0$ at the EW fixed point—implying there is a phase transition between rough ($\chi > 0$) and smooth ($\chi < 0$) interfaces. It is important to reiterate that the nonlinearity in the KPZ equation is relevant in $d = 1$ and $d = 2$, as well in $d = 3$ when $\lambda > \lambda_*^{\mathrm{c}}$. In such cases, interfacial dynamics fall into the KPZ universality class.

Though the KPZ equation was developed to explain the dynamic scaling growing interfaces, the resultant universality class emerges in unrelated systems. For example, stochastic particle dynamics models of interfaces—including the ballistic deposition (BD) and corner growth (CG) models—exhibit KPZ universality [11]. By taking the gradient of the KPZ equation (10) and expressing the result in terms of $\boldsymbol{v}(\boldsymbol{x},t) := -\lambda \boldsymbol{\nabla} h(\boldsymbol{x},t)$, we arrive at the noisy Burgers equation: a hydrodynamical description of shock waves in a fluid [6]. If, on the other hand, we employ a Cole–Hopf transformation and express the KPZ equation (10) in terms of $W(\boldsymbol{x},t) := \exp\{\lambda h(\boldsymbol{x},t)/(2\nu)\}$, we arrive at a description of a directed polymer in a random potential [6]. Interestingly, a recent study found KPZ universality in quantum entanglement entropy growth [12], and in doing so drew a quantum mechanical analogy to the aforementioned stochastic particle dynamics, hydrodynamics, and directed polymer descriptions. Such widespread examples demonstrate the profound impact of the KPZ equation and universality class on modern statistical physics.

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